CONFORMALLY PARALLEL G_2 STRUCTURES ON A CLASS OF SOLVMANIFOLDS

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Abstract. Starting from a 6-dimensional nilpotent Lie group N endowed with an invariant $SU(3)$ structure, we construct a homogeneous conformally parallel G_2-metric on an associated solvmanifold. We classify all half-flat $SU(3)$ structures that endow the rank-one solvable extension of N with a conformally parallel G_2 structure. By suitably deforming the $SU(3)$ structures obtained, we are able to describe the corresponding non-homogeneous Ricci-flat metrics with holonomy contained in G_2. In the process we also find a new metric with exceptional holonomy.

1. Introduction

A seven-dimensional Riemannian manifold (Y, g) is called a G_2-manifold if it admits a reduction of the structure group of the tangent bundle to the exceptional Lie group G_2. The presence of a G_2 structure is equivalent to the existence of a certain type of three-form φ on the manifold. Whenever this 3-form is covariantly constant with respect to the Levi–Civita connection then the holonomy group is contained in G_2, and the corresponding manifold is called parallel. The development of the theory of explicit metrics with holonomy G_2 follows the by-now-classical line of Bonan [5], Fernández and Gray [15], Bryant [7] and Salamon [9]. We shall review a few relevant facts in section 2.

Interesting non-compact examples are provided by Gibbons, Lü, Pope, Stelle in [17], where incomplete Ricci-flat metrics of holonomy G_2 with a 2-step nilpotent isometry group N acting on orbits of codimension one are presented. It turns out that these metrics have scaling symmetries generated by a homothetic Killing vector field, and are locally isometric (modulo a conformal change) to homogeneous metrics on solvable Lie groups. The solvable Lie group in question is obtained by extending the isometry group of the original manifold, and can be seen as the universal cover of the product of $\mathbb{R}$ with the 2-step nilmanifold corresponding to N, which is a compact quotient $\Gamma\backslash N$ by a discrete uniform subgroup. Solvmanifolds — that is solvable Lie groups endowed with a left-invariant metric — and in particular solvable extensions of nilpotent Lie groups provide instances of homogeneous Einstein manifolds. The fact that any nilpotent Lie algebra of dimension 6 admits a solvable extension carrying Einstein metrics [31] will be of the foremost importance.

We shall concentrate on *conformally parallel G_2 structures*, characterised by the fact that the Riemannian metric g can be modified to metric with holonomy a subgroup of G_2 by a transformation

$$g \mapsto e^{2f} g,$$

for some function f.

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In the light of [31], it is natural to study such G_2 structures on a rank-one solvable extension of a metric 6-dimensional nilpotent Lie algebra $\mathfrak{n}$ endowed with an $SU(3)$ structure (ω, ψ^+) and a non-singular self-adjoint derivation D which is diagonalisable by a unitary basis. This last condition is equivalent to $(DJ)^2 = (JD)^2$ and we show that this is the compatibility that one has to impose between D and the $SU(3)$ structure in order to obtain the non-compact examples found in [17].

As shown in section 3, such an extension is given by a metric Lie algebra $\mathfrak{s} = \mathfrak{n} \oplus \mathbb{R}H$ with bracket

$$[H, U] = DU, \quad [U, V] = [U, V]_{\mathfrak{n}\times\mathfrak{n}},$$

where $U, V \in \mathfrak{n}$ and $H \perp \mathfrak{n}, \|H\| = 1$. The subscript denotes the Lie bracket on $\mathfrak{n}$, and the inner product extends that of $\mathfrak{n}$. There is a natural G_2 structure on the manifold $Y = N \times \mathbb{R}$ corresponding to the 3-form

$$\varphi = \omega \wedge H^\flat + \psi^+ \in \Lambda^3 T^*Y,$$

where $\flat$ is the isomorphism of T onto T^* induced by the metric. The Lie algebra $\mathfrak{s}$ is isomorphic to each fibre of the principal fibration $T^*Y \longrightarrow Y$, and we prove the

Main result. (Y, φ) *is conformally parallel if and only if* $\mathfrak{n}$ *is either* $\mathbb{R}^6$*, or* 2*-step nilpotent but not isomorphic to the Lie algebra* $\mathfrak{h}_3 \oplus \mathfrak{h}_3$*,*

where $\mathfrak{h}_3$ denotes the real 3-dimensional Heinsenberg algebra (cf. §4). The operator D has the same eigenvalue type of the derivation considered by Will to construct Einstein metrics on 7-dimensional solvmanifolds [31].

In section 5 we describe explicitly the corresponding metrics g with holonomy a non-trivial subgroup of G_2. Half of such metrics have $Hol(g) = G_2$ and stem from the three irreducible 2-step nilpotent Lie algebras. The remaining metrics have holonomy either $SU(2)$ or $SU(3)$ and correspond to Lie algebras with abelian summands. Using this we show that some metrics have also been considered by [17] in the study of special domain walls in string theory. We are able to produce a new metric with holonomy equal to G_2, that arises from the 6-dimensional Lie algebra spanned by $e^1, \dots, e^6$ with

$$e_2 = [e_5, e_4], \quad e_3 = [e_6, e_4] = [e_1, e_5]$$

as the only non-trivial brackets.

The conformally parallel G_2 structure forces the initial $SU(3)$ structure to be of a special kind, known in the literature as half-flat [11]. This turns out to be a useful notion, which allows one to find explicit metrics with holonomy G_2 by investigating the corresponding Hitchin flow [21]. Section 6 is especially devoted to such a description. We determine a solution of the evolutions equations and compare the resulting G_2 holonomy metrics with the ones previously described. These rank-one solvmanifolds S admit then a pair of distinguished metrics. The first is the homogeneous Einstein metric with negative scalar curvature constructed in [31]. The other arises by conformally changing a homogeneous metric and possesses a homothetic Killing field, i.e. a vector field with respect to which the Lie derivative of g is a multiple of the identity; our investigation proves that it is also obtainable by evolving the original $SU(3)$ structure.

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2. G structures in 6 and 7 dimensions

Suppose that X indicates a six-dimensional nilmanifold with an invariant almost Hermitian structure. Thus, X is endowed with an orthogonal almost complex structure J and a non-degenerate 2-form ω which induce a Riemannian metric h. An $SU(3)$-reduction of the structure group is determined by fixing a real 3-form ψ^+ lying in the S^1-bundle of unit elements inside the canonical bundle $[\![\Lambda^{3,0}]\!]$ at each point. We adopt the parenthetical notation of [30] to indicate real modules of $\{p,q\}$-forms underlying the complex space $\Lambda^{p+q}_{\mathbb{C}}$. Let $\Psi = \psi^+ + i\psi^-$ be the associated holomorphic section (so that $J\psi^- = -\psi^+$). The description is always intended to be local, so one can define the forms

$$\begin{aligned} \omega &= e^{14} - e^{23} + e^{56}, \\ \psi^+ + i\psi^- &= (e^1 + ie^4) \wedge (e^2 - ie^3) \wedge (e^5 + ie^6), \end{aligned} \tag{2.1}$$

of type $(1,1)$ and $(3,0)$ relative to J. It has become customary to suppress wedge signs when writing differential forms, so $e^{ij\cdots}$ indicates $e^i \wedge e^j \wedge \ldots$ from now on. Following [11] and [6] we tackle six-dimensional geometry by means of the enhanced Gray and Hervella decomposition of the intrinsic torsion space into five representations $\mathcal{W}_1, \ldots, \mathcal{W}_5$. These are the $SU(3)$-modules appearing in $\Lambda^1 \otimes \big([\![\Lambda^{2,0}]\!] \oplus \mathbb{R}\big)$ that identify the kind of almost Hermitian structure. Complex $SU(3)$-manifolds are for instance characterised by the vanishing of the intrinsic torsion components belonging to $\mathcal{W}_1 \cong \mathbb{R} \oplus \mathbb{R}, \mathcal{W}_2 \cong \mathfrak{su}(3) \oplus \mathfrak{su}(3)$. Tagging the irreducible 'halves' by $\pm$, one can correspondingly split the Nijenhius tensor $N_J = N_J^+ + N_J^-$. The modules $\mathcal{W}_1^\pm, \mathcal{W}_2^\pm$ can be defined explicitly by prescribing the various types of the real forms

$$\begin{aligned} d\psi^+ &= -2W_5 \wedge \psi^+ + W_2^+ \wedge \omega + W_1^+ \omega^2 \\ d\psi^- &= 2W_5 \wedge \psi^- + W_2^- \wedge \omega + W_1^- \omega^2 \end{aligned}$$

corresponding to $\Lambda^4 T^*X \cong [\![\Lambda^{0,1}]\!] \oplus [\Lambda_0^{1,1}] \oplus \mathbb{R}$. The nought in the middle term denotes $(1,1)$-forms α satisfying $\alpha \wedge \omega = 0$, called primitive.

Moving up one dimension, we consider a product Y of X with $\mathbb{R}$, endowed with metric g. Indicating by e^7 the unit 1-form on the real line one obtains a basis for the cotangent spaces T_y^*Y. The manifold Y inherits a non-degenerate three-form $\varphi = \omega \wedge e^7 + \psi^+$ which is stable, *à la* Hitchin [21], and defines a reduction to the exceptional group. The fundamental material for the G_2 story can be found in standard references [30, 23]. Let us only recall that the Riemannian geometry of Y is completely determined by the tensor

$$\varphi = e^{125} - e^{345} + e^{567} + e^{136} + e^{246} - e^{237} + e^{147}.$$

The seminal results of Fernández and Gray [15] permit one to describe G_2 geometry exclusively in algebraic terms, by looking at the various components of $d\varphi, d{*\varphi}$ in the irreducible summands $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ and $\mathcal{X}_4$ of the space $T^*Y \otimes \mathfrak{g}_2^\perp$. Many authors have studied special classes of G_2 structures, see for instance [10, 16, 12]. Before concentrating on a particular situation, recall that in general the exterior derivatives can be expressed as

$$\left\{ \begin{array}{rcl} d{*\varphi} &=& 4\tau_4 \wedge *\varphi + \tau_2 \wedge \varphi \\ d\varphi &=& \tau_1 \wedge *\varphi + 3\tau_4 \wedge \varphi + *\tau_3 \end{array} \right. ,$$

where the various τ_i's represent the differential forms corresponding to the representations $\mathcal{X}_i$, as in [8]. For example τ_4 is the 1-form encoding the 'conformal' data of the structure. With the convention of dropping all unnecessary wedge signs, the torsion three-form of the unique G_2-connection [16] is given by

$$\Phi = \tfrac{7}{6}\tau_1\varphi - {*d\varphi} + {*(4\tau_4\,\varphi)}$$

in terms of $\tau_1 = \frac{1}{7}g(d\varphi, *\varphi)$ and $\tau_4 = -\frac{3}{4}{*(*d\varphi \wedge \varphi)}$, the latter being the Lee form of the 7-manifold, essentially.

Our aim is to study conformally parallel G_2 structures on Riemannian products, otherwise said manifolds $X \times \mathbb{R}$ whose intrinsic torsion belongs to the class $\mathcal{X}_4$ only. If this is the case, the above pair of equations simplifies to

$$d{*\varphi} = 4\tau_4 \wedge *\varphi, \qquad d\varphi = 3\tau_4 \wedge \varphi$$

and the obstruction to the reduction of the holonomy can be written as $\Phi = *(\tau_4\,\varphi)$, proportional to the Hodge dual of $d\varphi$. Now τ_4 is a closed 1-form in the more general setting of G_2T-structures, so as soon as one has $\dim H^1(Y,\mathbb{R}) = 1$ (see (3.3)), it will be natural to assume it is proportional to e^7. So let us rewrite those relations as

$$\left\{\begin{array}{rcl} d{*\varphi} & = & 4me^7 \wedge *\varphi \\ d\varphi & = & 3me^7 \wedge \varphi \end{array}\right. , \tag{2.2}$$

which also serve as a definition for the real constant m. To prevent the holonomy of the metric g from reducing to G_2, we implicitly assume that m does not vanish.

We shall next fit the geometric picture into the theory of Lie algebras, and suppose X is a nilpotent Lie group. This is indeed no real restriction since [32] any Riemannian manifold X admitting a transitive nilpotent Lie group of isometries is essentially a nilpotent Lie group N with an invariant metric. We shall determine which six-dimensional (1-connected) nilpotent Lie groups N generate conformally parallel structures on manifolds of a special kind, described hereby.

3. Solvable extensions of nilpotent Lie algebras

Let (N, h) denote a six-dimensional connected and simply-connected nilpotent Lie group with a left-invariant Riemannian metric, and $\mathfrak{n}$ its Lie algebra. The orthonormal basis $\{e^1, \dots, e^6\}$ of the cotangent bundle T^*N is intended to be nilpotent, i.e. such that $de^i \in \Lambda^2 V_{i-1}$, where the spaces $V_j = \mathrm{span}_{\mathbb{R}}\{e^1, \dots, e^{j-1}\}$ filtrate the dual Lie algebra: $0 \subset V_1 \subset \dots \subset V_5 \subset V_6 = \mathfrak{n}^*$. The step-length of $\mathfrak{n}$ is defined as the number p of non-zero subspaces appearing in the lower central series

$$\mathfrak{n} \supseteq [\mathfrak{n}, \mathfrak{n}] \supseteq \big[[\mathfrak{n}, \mathfrak{n}], \mathfrak{n}\big] \supseteq \dots \supseteq \{0\}.$$

Given this, the terms Abelian and 1-step are synonymous. We shall need later the fact [29] that a nilmanifold $\Gamma\backslash N$ and the Lie algebra of its universal cover have isomorphic cohomology theories, $H^*(\mathfrak{n}) \cong H^*_{\mathrm{dR}}(\Gamma\backslash N)$.

Fix now a unit element $H \notin \mathfrak{n}$ and suppose there exists a non-singular self-adjoint derivation D of $\mathfrak{n}$ endowing

$$\mathfrak{s} = \mathfrak{n} \oplus \mathbb{R}H \tag{3.1}$$

with the structure of a solvable Lie algebra. In other words think of $\mathfrak{s}$ as an extension of the following kind

3.1. **Definition.** A metric solvable Lie algebra $\big(\mathfrak{s}, \langle\ ,\ \rangle\big)$ is said *of Iwasawa type* if

(1) $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$, with $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ and $\mathfrak{a} = \mathfrak{n}^{\perp}$ Abelian;
(2) ad_H is self-adjoint with respect to the scalar product $\langle\ ,\ \rangle$ and non-zero, for all $H \in \mathfrak{a}, H \neq 0$;
(3) for some (canonical) element $\tilde{H} \in \mathfrak{a}$, the restriction of $ad_{\tilde{H}}$ to $\mathfrak{n}$ is positive-definite.

The terminology is clearly reminiscent of the Iwasawa decomposition of a semisimple Lie group. This is indeed no coincidence, for any irreducible symmetric space of non-compact type $Y = G/K$ can be isometrically identified with the solvmanifold $S = AN$ relative to the decomposition $G = KAN$ of the connected component of the isometry group of Y. Iwasawa-type extensions are instances of standard solvmanifolds in the sense of Heber, and in a way represent the basic model of standard Einstein manifolds [20]. Now the nilpotent Lie groups of concern (actually all, up to dimension six) always admit Einstein solvable extensions [26], yet we wish to stress that all known examples of non-compact homogeneous spaces with Einstein metrics are of this kind, modulo isometries. What is more, they are completely solvable, i.e. the eigenvalues of any inner derivation are real. The curvature of these spaces must be non-positive, because Ricci-flat homogeneous manifolds are flat [2], and Alekseevskiĭ has conjectured that a non-compact homogeneous Einstein manifold has a transitive solvable isometry group. The latter cannot be unimodular, as the space is assumed to be non-flat [14]. This is in contrast to the nilpotent picture, where a cocompact discrete subgroup always exists [28], under the hypothesis of rationality of the structure constants.

The whole point of reducing to rank one is that in the Einstein case, this is no big specialisation, for [20] classifying standard Einstein solvmanifolds is essentially the same as determining those with $codim\,[\mathfrak{s}, \mathfrak{s}] = 1$.

Since N has an invariant $SU(3)$ structure one can suppose there exists a diagonalisable operator $D \in Der(\mathfrak{n})$ with respect to a Hermitian basis, that determines the rank-one extension as in (3.1). That entails that there is indeed a unitary basis consisting of eigenvectors — let us still call it $\{e_i\}, i = 1 \dots 6$ — for which the matrix associated to $D = ad_{\tilde{H}}$ is diagonal. Hence,

$$\mathrm{ad}_{\tilde{H}}(e_i) = c_i e_i \tag{3.2}$$

for some real constants c_i, which must be positive in order to satisfy Definition 3.1. The derivation D is chosen to be precisely ad_{e_7}, and since the Cartan subalgebra $\mathfrak{a}$ is now one-dimensional the only inner automorphism acting on $\mathfrak{n}$ is the bracket with the vector $\tilde{H} = e_7$, which is self-adjoint for the inner product, and non-degenerate because $c_j \neq 0$, for all j's. Therefore, the Maurer-Cartan equations of the rank-one solvable extension $\mathfrak{s} = \mathfrak{n} \oplus \mathbb{R}e_7$ assume the form

$$\begin{cases} de^j = \hat{d}e^j + c_j e^{j7}, & 1 \leqslant j \leqslant 6 \\ de^7 = 0, \end{cases} \tag{3.3}$$

where the 'hat' indicates derivatives relative to the six-dimensional world, i.e. $\hat{d} = d|_{\Lambda^*\mathbb{R}^6}$, and $\{e^j\}$ is the basis of $\mathfrak{s}$ dual to $\{e_i\}$. Results of Heber and Will [20, 31] guarantee that $Y = N \times \mathbb{R}$ admits Einstein metrics, in fact there exists a unique choice of the vector $(c_1, \dots, c_6)$ such that the inner product $\langle\, , \rangle$ is Einstein.

In general, the Lie structure of $\mathfrak{n}$ is defined by

$$\begin{cases} \hat{d}e^1 &= a_1 e^{12} + \dots + a_{15} e^{56} \\ \hat{d}e^2 &= a_{16} e^{12} + \dots + a_{30} e^{56} \\ \dots\dots\dots\dots\dots\dots \\ \hat{d}e^6 &= a_{76} e^{12} + \dots + a_{90} e^{56} \end{cases} \tag{3.4}$$

where all coefficients a_k are real numbers.

Remark 3.2. This is a good point to see that the existence of a unitary basis diagonalising $ad_{e_7} = D$ is tantamount to requiring that D and JDJ commute, or $(DJ)^2 = (JD)^2$. This follows directly from (2.1), (3.2), and the computation for e_1 is heuristic

$$DJe_1 = c_4 e_4 = \tfrac{c_4}{c_1} JDe_1, \quad \text{hence}$$
$$(DJ)^2 e_1 = \tfrac{c_4}{c_1} DJ(JDe_1) = -\tfrac{c_4}{c_1} D^2 e_1 = -c_4 c_1 e_1 = \\ -\tfrac{c_1}{c_4} J(c_4^2 e_4) = -\tfrac{c_1}{c_4} JD^2 e_4 = \tfrac{c_1}{c_4} JDDJe_1 = (JD)^2 e_1.$$

It has to be noticed though that J need not necessarily be an almost complex structure for the argument. In fact, any endomorphism $\mathcal{I}$ of the tangent bundle of N such that $\mathcal{I}\langle e_1\rangle = \langle e_4\rangle$, $\mathcal{I}\langle e_4\rangle = \langle e_1\rangle$ *et cetera* does the job, since then $\mathcal{I}^{-1}D\mathcal{I}$ and D are simultaneously diagonalisable, hence commute.

4. The classification

Before we start investigating equations (2.2) in relation to the induced geometry on N^6, let us discuss the delicate point of the choice of the $SU(3)$ reduction. Define $\psi^\pm$ as in (2.1) with $\{e_i\}$ a unitary basis that diagonalises D. A reduction to $SU(3)$ is determined by the choice of an element

$$\tilde{\psi}^+ = \psi^+ \cos\theta + \psi^- \sin\theta$$

(for some angle θ) in the circle generated by ψ^+ and ψ^- in $\mathcal{W}_1$. In general, it is impossible to express $\tilde{\psi}^+$ in terms of a basis that diagonalises D as simply as in (2.1). The proof of next theorem shows that one can in fact assume that $\theta = 0$.

Moreover, one can say is that there is a unique — up to sign — closed 3-form in the circle (Proposition 4.6).

Let us write

$$\varphi = \omega e^7 + \psi^+, \qquad *\varphi = \psi^- e^7 + \tfrac{1}{2}\omega^2,$$

whence one immediately finds that

$$d\omega e^7 + d\psi^+ = -3m\psi^+ e^7, \qquad d\psi^- e^7 + \omega d\omega = 2m\omega^2 e^7. \tag{4.1}$$

Reflecting the splitting of the fibres of the cotangent bundle $T_y^*Y = \mathbb{R}^6 \oplus \mathbb{R}e^7$, the relations (3.3) give

$$d\omega = \hat{d}\omega - \big((c_1 + c_4)e^{14} - (c_3 + c_2)e^{23} + (c_5 + c_6)e^{56}\big)e^7,$$

so

$$d\omega^2 = \hat{d}\omega^2 + 2\big((c_1+c_4+c_3+c_2)e^{1423} + (c_3+c_2+c_5+c_6)e^{2356} - (c_1+c_4+c_5+c_6)e^{1456}\big)e^7.$$

Similarly one computes the exterior derivatives of the real 3-forms:

$$\begin{aligned} d\psi^+ &= \hat{d}\psi^+ + (c_1+c_2+c_5)e^{1257} + (c_1+c_3+c_6)e^{1367} \\ &\quad - (c_3+c_4+c_5)e^{3457} + (c_2+c_4+c_6)e^{2467}, \\ d\psi^- &= \hat{d}\psi^- + (c_1+c_2+c_6)e^{1267} - (c_1+c_3+c_5)e^{1357} \\ &\quad - (c_2+c_4+c_5)e^{2457} - (c_3+c_4+c_6)e^{3467}. \end{aligned}$$

When, in general, G_2-manifolds Y are constructed starting from six dimensions, many of their features are determined by the underlying $SU(3)$ structure, and the following definition becomes natural

4.1. **Definition.** [11] An almost Hermitian manifold is *half-flat*, or *half-integrable*, if the reduction is such that both ψ^+ and ω^2 are closed (with respect to $\hat{d}$).

This is the same as asking that the intrinsic torsion components W_1^+, W_2^+, W_4 and W_5 vanish simultaneously. This sort of structure appears, in various disguises, on any hypersurface in $\mathbb{R}^7$ (or Joyce manifold, for that matter), and its possible rôle in $\mathcal{M}$-theory has been recently examined [19, 4].

Plugging the previous equations into system (2.2) allows us to discover a geometrical constraint, for

4.2. **Lemma.** *When (Y,φ) is conformal to a G_2-holonomy manifold, N has a half-flat $SU(3)$ structure.*

Proof. This is clear if one considers the terms in (4.1) that belong to $(e^7)^\perp$. □

On the other hand, the components of (2.2) in the direction of e^7 read

$$\left\{\begin{array}{rcl} \hat{d}\omega &=& -(c_1+c_2+c_5+3m)e^{125} - (c_1+c_3+c_6+3m)e^{136} - \\ && \quad (c_3+c_4+c_5+3m)e^{345} - (c_2+c_4+c_6+3m)e^{246} \\ \hat{d}\psi^- &=& (2m-c_1-c_2-c_3-c_4)e^{1423} + (2m-c_3-c_2-c_5-c_6)e^{2356} + \\ && (2m+c_1+c_4+c_5+c_6)e^{1456}. \end{array}\right. \tag{4.2}$$

We will show that the derivation $D = ad_{e_7}$ has an eigenvector (for instance e_1) belonging to $[\mathfrak{n},\mathfrak{n}]^\perp$, so the structure of $\mathfrak{s}$ is determined by equations (3.3), with $\hat{d}e^1 = 0$ and $\hat{d}e^j$ given by (3.4) for $j = 2,\dots,6$. The point is to find all possible coefficients $a_k, k = 16,\dots,90$ and $c_j, 1 \leqslant j \leqslant 6$ such that $d^2(e^j) = 0$ and (4.1) are satisfied, for some non-vanishing m. In this way we obtain the following classifying result

4.3. **Theorem.** *Let N be a nilpotent Lie group of dimension 6 endowed with an invariant $SU(3)$ structure $(\omega, \tilde{\psi}^+)$. Suppose there is a non-singular and self-adjoint derivation D of the Lie algebra $\mathfrak{n}$ such that $(DJ)^2 = (JD)^2$. Then on the solvable extension $\mathfrak{s} = \mathfrak{n} \oplus \mathbb{R}e_7$ with $\mathrm{ad}_{e_7} = D$, the G_2 structure $\varphi = \omega \wedge e^7 + \tilde{\psi}^+$ is conformally parallel if and only if $\mathfrak{n}$ is isomorphic to one of the following:*

$$
\begin{array}{ll}
(0,0,0,e^{12},e^{13},e^{23}), & (0,0,0,0,e^{12},e^{13}),\\
(0,0,0,0,e^{12},e^{14}+e^{23}), & (0,0,0,0,0,e^{12}+e^{34}),\\
(0,0,0,0,e^{13}+e^{42},e^{12}+e^{34}), & (0,0,0,0,0,e^{12}),\\
(0,0,0,0,0,0). &
\end{array}
$$

Though the list does not appear that meaningful at first sight, it becomes more significant once considered in relation to the descriptions given in [27, 18].

Proof. Since D is a derivation, it must preserve the orthogonal splitting $[\mathfrak{n},\mathfrak{n}]\oplus[\mathfrak{n},\mathfrak{n}]^{\perp}$. Indicating the derived algebra $[\mathfrak{n},\mathfrak{n}]$ by $\mathfrak{n}^1$, one infers that $ad_{e_7}(\mathfrak{n}^1)\subseteq\mathfrak{n}^1$, hence $ad_{e_7}(\mathfrak{n}^1)^{\perp}\subseteq(\mathfrak{n}^1)^{\perp}$. Then one can suppose that there exists a unitary basis $\{e^i\}$ which diagonalises D with e^1 closed in $\mathfrak{n}^*$. The structure equations of $\mathfrak{s}$ are given by (3.3) and (3.4), with $a_j=0$ for all $j=1,\dots,15$ and the 3-form $\tilde\psi^+$ can be then expressed, in terms of the previous basis $\{e^i\}$, as

$$\psi^+\cos\theta+\psi^-\sin\theta,$$

for some angle θ, where $\psi^{\pm}$ are given by (2.1). It is necessary to impose the quadratic relations $d^2e^i=0$ together with the linear equations (2.2) , for a total of $35\cdot5+56$ constraints. The complete system has $75+6+1+1$ unknown variables a_k,c_j,m,θ. Given the number of parameters and equations, the results were also checked with the MAPLE package. Inserting the coordinates, (2.2) yields a bulk of 56 linear constraints on 83 coefficients. We have to distinguish two cases: $\theta=0$ and $\theta\neq0$. If θ is not zero, the imposition of $d^2e^i=0$ and $c_l\neq0$ gives no solution.

If $\theta=0$ the equations (2.2) reduce to (4.2). Replacing the respective expressions in (3.4), one gets

$$
\begin{aligned}
de^1 &= c_1e^{17},\\
de^2 &= c_2e^{27}+a_{16}e^{12}+(a_{65}-a_{22}-a_{89})e^{13}+a_{55}e^{14}+(a_{77}+a_{56})e^{15}+a_{20}e^{16}+\\
&\quad b_1e^{23}+a_{22}e^{24}+a_{23}e^{25}+a_{24}e^{26}+(a_{88}-a_{32}-a_{48}-a_{64})e^{34}+a_{26}e^{35}+\\
&\quad a_{27}e^{36}+(2a_{62}+2c_1-2c_5+a_{53}+a_{20}+a_{67}-a_{76})e^{45}+a_{29}e^{46}+a_{30}e^{56},\\
de^3 &= c_3e^{37}+(-a_{22}+a_{89}-a_{65})e^{12}+a_{32}e^{13}+a_{33}e^{14}+\\
&\quad (4c_1-2c_2-2c_5+2a_{53}+3a_{62}+3a_{20}-a_{76})e^{15}+a_{35}e^{16}+a_{36}e^{23}+\\
&\quad (-a_{88}+a_{16}+a_{48}+a_{64})e^{24}+(-a_{26}+2a_{24}+a_{66})e^{25}+a_{39}e^{26}-a_{22}e^{34}+\\
&\quad b_2e^{35}+b_3e^{36}+(a_{29}-a_{77}-a_{56}-a_{35})e^{45}+\\
&\quad (5a_{20}+6c_1-4c_2-4c_5+4a_{53}+6a_{62}-a_{76}+2c_3+a_{85}-a_{57})e^{46}+a_{45}e^{56},\\
de^4 &= c_4e^{47}+a_{55}e^{12}+a_{33}e^{13}+a_{48}e^{14}+a_{49}e^{15}+a_{50}e^{16}+\\
&\quad (-a_{16}-a_{32})e^{23}-a_{33}e^{24}+a_{53}e^{25}+a_{56}e^{26}+a_{55}e^{34}+\\
&\quad a_{56}e^{35}+a_{57}e^{36}-a_{50}e^{45}+a_{49}e^{46}+(-a_{88}+2a_{64}-a_{74})e^{56},\\
de^5 &= c_5e^{57}+(a_{35}+a_{56})e^{12}+a_{62}e^{13}+a_{49}e^{14}+a_{64}e^{15}+a_{65}e^{16}+a_{66}e^{23}+a_{67}e^{24}+\\
&\quad b_4e^{25}+(-a_{83}+2a_{71}-a_{30})e^{26}+a_{29}e^{34}+a_{71}e^{35}+a_{72}e^{36}-a_{89}e^{45}+\\
&\quad a_{74}e^{46}+b_5e^{56},
\end{aligned}
$$

$$\begin{aligned}de^6 &= c_6e^{67} + a_{76}e^{12} + a_{77}e^{13} + a_{50}e^{14} + (2a_{89} - a_{65})e^{15} + b_6e^{16}+\\ &\quad (2a_{23} + a_{27} + a_{39})e^{23} + (a_{29} - a_{77} - a_{56} - a_{35})e^{24} + a_{83}e^{25} + b_7e^{26}+\\ &\quad a_{85}e^{34} + (2a_{33} - 2a_{36} + a_{72} + a_{45})e^{35} + b_8e^{36} + a_{88}e^{45} + a_{89}e^{46} + b_9e^{56},\\ de^7 &= 0,\end{aligned}$$

in terms of a certain number of parameters. We have put

$$b_1 = -a_{83} + a_{55} + a_{71}, \qquad b_2 = a_{27} + a_{39} + a_{23}, \qquad b_3 = -a_{24} - a_{66},$$
$$b_4 = -a_{33} - a_{72} + a_{36} - a_{45}, \quad b_5 = -a_{49} + a_{24} - a_{26} + a_{66}, \quad b_6 = -a_{88} + a_{64} - a_{74},$$
$$b_7 = a_{33} + a_{72} - a_{36}, \qquad b_8 = -a_{71} + a_{30}, \qquad b_9 = a_{39} + a_{23} - a_{50}$$

for convenience. Besides, the following relations must hold:

$$c_4 = -5c_1 + 3c_2 + 4c_5 - 4a_{53} - 5a_{62} - 4a_{20} - c_3 - a_{67} + a_{76} - a_{85},$$
$$c_6 = 3c_2 - c_3 + 3c_5 + a_{57} - 3a_{53} - 4c_1 - 4a_{62} - 4a_{20},$$
$$m = c_1 - c_2 - c_5 + a_{53} + a_{62} + a_{20}.$$

Only at this point it seems realistic to annihilate the quadratic relations coming from the Jacobi identity, hence set to zero the coefficients of the terms e^{ij7} appearing in the various $d^2e^i = 0$. Since $c_j \neq 0$ for all $j = 1, \ldots, 6$, the closure of de^i kills all b_l's above, and furthermore

$$a_{16} = a_{22} = a_{24} = a_{23} = a_{32} = a_{33} = a_{36} = a_{48} = a_{49} = a_{50} = a_{55} = a_{64} = a_{71} = a_{89} = 0.$$

Thus, the structure equations eventually reduce to a simpler form

$$\begin{aligned}de^1 &= c_1e^{17}\\ de^2 &= c_2e^{27} + a_{65}e^{13} + (a_{77} + a_{56})e^{15} + a_{20}e^{16} - a_{74}e^{34}+\\ &\quad (2a_{62} + 2c_1 - 2c_5 + a_{53} + a_{20} + a_{67} - a_{76})e^{45} + a_{29}e^{46}\\ de^3 &= c_3e^{37} - a_{65}e^{12} + (4c_1 - 2c_2 - 2c_5 + 2a_{53} + 3a_{62} + 3a_{20} - a_{76})e^{15}+\\ &\quad a_{35}e^{16} + a_{74}e^{24} + (a_{29} - a_{77} - a_{56} - a_{35})e^{45}+\\ &\quad (5a_{20} + 6c_1 - 4c_2 - 4c_5 + 4a_{53} + 6a_{62} - a_{76} + 2c_3 + a_{85} - a_{57})e^{46}\\ de^4 &= c_4e^{47} + a_{53}e^{25} + a_{56}e^{26} + a_{56}e^{35} + a_{57}e^{36}\\ de^5 &= c_5e^{57} + (a_{35} + a_{56})e^{12} + a_{62}e^{13} + a_{65}e^{16} + a_{67}e^{24} + a_{29}e^{34} + a_{74}e^{46}\\ de^6 &= c_6e^{67} + a_{76}e^{12} + a_{77}e^{13} - a_{65}e^{15} + (a_{29} - a_{77} - a_{56} - a_{35})e^{24}+\\ &\quad a_{85}e^{34} - a_{74}e^{45}\\ de^7 &= 0.\end{aligned}$$

In particular, the vanishing of the coefficients of e^{137}, e^{347} in $d^2(e^2)$ and of e^{127}, e^{247} in $d^2(e^3)$ yields

$$a_{65}(c_1 - c_2 + c_3) = 0, \quad a_{74}(-c_2 - c_3 + c_4) = 0,$$
$$a_{65}(-c_1 - c_2 + c_3) = 0, \quad a_{74}(c_2 - c_3 + c_4) = 0,$$

thus $a_{65} = a_{74} = 0$. The terms e^{357}, e^{267} in $d^2(e^4)$ similarly give

$$a_{56}(-c_3 + c_4 - c_5) = 0, \quad a_{56}(-c_2 + c_4 - c_6) = 0, \quad a_{53}(-c_2 + c_4 - c_5) = 0,$$
$$a_{57}(-c_2 + a_{57} + a_{53} + c_1 + a_{62} + c_3 + a_{67} - a_{76} + a_{85}) = 0.$$

Altogether, the following cases crop up. We shall examine them one by one trying to make further coefficients disappear.

Case a) $a_{53} = a_{56} = a_{57} = 0$ (corresponding to $\hat{d}e^4 = 0$). The relation $d^2 = 0$ yields $a_{29} = a_{35} = a_{77} = 0$, and six non-isomorphic algebra types come out:

$$(-me^{17}, -me^{27}, -me^{37}, -me^{47}, -me^{57}, -me^{67}, 0) \tag{4.3}$$

has an underlying Abelian Lie algebra, if one disregards the D-action. Next,

$$(-\tfrac{2}{3}me^{17}, -me^{27}, -\tfrac{4}{3}me^{37} + \tfrac{2}{3}me^{15}, -me^{47}, -\tfrac{2}{3}me^{57}, -me^{67}, 0) \tag{4.4}$$

extends $\mathfrak{n} \cong (0, 0, 0, 0, 0, e^{12})$;

$$\big(-\tfrac{3}{4}me^{17}, -me^{27}, -\tfrac{3}{2}me^{37} + \tfrac{1}{2}m(e^{15} - e^{46}), -\tfrac{3}{4}me^{47}, -\tfrac{3}{4}me^{57}, -\tfrac{3}{4}me^{67}, 0\big) \tag{4.5}$$

is clearly given by $\mathfrak{n} \cong (0, 0, 0, 0, 0, e^{12} + e^{34})$;

$$\big(-\tfrac{4}{5}me^{17}, -\tfrac{6}{5}me^{27} - \tfrac{2}{5}me^{45}, -\tfrac{7}{5}me^{37} + \tfrac{2}{5}m(e^{15} - e^{46}), -\tfrac{3}{5}me^{47}, -\tfrac{3}{5}me^{57}, -\tfrac{4}{5}me^{67}, 0\big) \tag{4.6}$$

attached to $\mathfrak{n} \cong (0, 0, 0, 0, e^{12}, e^{14} + e^{23})$;

$$(-me^{17}, -\tfrac{5}{4}me^{27} - \tfrac{1}{2}me^{45}, -\tfrac{5}{4}me^{37} - \tfrac{1}{2}me^{46}, -\tfrac{1}{2}me^{47}, -\tfrac{3}{4}me^{57}, -\tfrac{3}{4}me^{67}, 0), \tag{4.7}$$

whose $\mathfrak{n}$ is essentially $(0, 0, 0, 0, e^{12}, e^{13})$;

$$\big(-\tfrac{2}{3}me^{17}, -\tfrac{4}{3}me^{27} - \tfrac{1}{3}m(e^{16} + e^{45}), -\tfrac{4}{3}me^{37} + \tfrac{1}{3}m(e^{15} - e^{46}), -\tfrac{2}{3}me^{47}, -\tfrac{2}{3}me^{57}, -\tfrac{2}{3}me^{67}, 0\big) \tag{4.8}$$

is an extension of the Iwasawa Lie algebra, isomorphic to $(0, 0, 0, 0, e^{13} + e^{42}, e^{12} + e^{34})$.

b) $a_{53} = a_{56} = 0, a_{57} = c_5 - c_1 - a_{62} - c_3 - a_{67} + a_{76} - a_{85}$. Up to isomorphism, one gets the two Lie algebras with structure (4.4) and (4.7).

c) $a_{56} = a_{57} = 0, c_2 = \frac{5}{2}c_1 - \frac{3}{2}c_5 + \frac{5}{2}a_{62} + 2a_{20} + \frac{1}{2}c_3 + \frac{1}{2}a_{67} + 2a_{53} - \frac{1}{2}a_{76} + \frac{1}{2}a_{85}$. This time around one finds the algebras of b) plus

$$(-\tfrac{3}{5}me^{17}, -\tfrac{3}{5}me^{27}, -\tfrac{6}{5}me^{37} + \tfrac{2}{5}me^{15}, -\tfrac{6}{5}me^{47} + \tfrac{2}{5}me^{25}, -\tfrac{3}{5}me^{57}, -\tfrac{6}{5}me^{67} + \tfrac{2}{5}me^{12}, 0), \tag{4.9}$$

which arises from $\mathfrak{n} \cong (0, 0, 0, e^{12}, e^{13}, e^{23})$.

d) $a_{56} = 0$, $c_2 = \frac{5}{2}c_1 - \frac{3}{2}c_5 + \frac{5}{2}a_{62} + 2a_{20} + \frac{1}{2}c_3 + \frac{1}{2}a_{67} + 2a_{53} - \frac{1}{2}a_{76} + \frac{1}{2}a_{85}$, $a_{57} = c_5 - a_{53} - c_1 - a_{62} - c_3 - a_{67} + a_{76} - a_{85}$, by which one regains (4.5).

e) $a_{53} = a_{57} = 0$, $a_{20} = -c_1 + c_2 + \frac{1}{2}c_5 - a_{62} - \frac{1}{2}c_3$, $a_{76} = c_1 + c_2 - c_5 + a_{62} + a_{67} + a_{85}$ yield no solutions.

f) $a_{76} = c_1 + c_2 - c_5 + a_{53} + a_{57} + a_{62} + a_{67} + a_{85}$, $c_3 = c_2$, $a_{20} = -c_1 + \frac{1}{2}c_2 + \frac{1}{2}c_5 - \frac{3}{4}a_{53} - a_{62} + \frac{1}{4}a_{57}$. The last case produces (4.5) one more time, and basically concludes the proof of the Theorem. □

The fact that so few Lie algebras are gotten may depend on the requirements made both on the G_2 structure and on the seven-dimensional construction. One easily recognizes that the groups associated to (4.3) and (4.4) are the torus T^6 and the product $T^3 \times \mathcal{H}^3$ of a torus with the real 3-dimensional Heisenberg group respectively, while (4.8) is attached to the complexified 3-dimensional Heisenberg group $\mathcal{H}_3^{\mathbb{C}}$.

In the case of an Einstein solvmanifold, the eigenvalues of the derivation D are positive integers $k_1 < \ldots < k_r$ without common divisors [20]. Indicating the respective multiplicities by $d_1, \ldots, d_r$, Heber defines the string $(k_1, \ldots, k_r; d_1, \ldots, d_r)$ the eigenvalue type of the solvmanifold. In our situation we have the following

4.4. **Corollary.** *With the above hypotheses, the eigenvalues of the derivation D have all the same sign. More precisely, the eigenvalue data of the algebras of Theorem 4.3 are given by the following scheme*

Nilpotent Lie algebra	*Eigenvalues*	*Multiplicities*
$(0,0,e^{15},0,0,0)$	$-\frac{2}{3}m,-m,-\frac{4}{3}m$	$2,3,1$
$(0,0,e^{15}+e^{64},0,0,0)$	$-\frac{3}{4}m,-m,-\frac{3}{2}m$	$4,1,1$
$(0,e^{45},e^{64}+e^{51},0,0,0)$	$-\frac{3}{5}m,-\frac{4}{5}m,-\frac{6}{5}m,-\frac{7}{5}m$	$2,2,1,1$
$(0,e^{45},e^{46},0,0,0)$	$-\frac{1}{2}m,-\frac{3}{4}m,-m,-\frac{5}{4}m$	$1,2,1,2$
$(0,e^{16}+e^{45},e^{15}+e^{64},0,0,0)$	$-\frac{2}{3}m,-\frac{4}{3}m$	$4,2$
$(0,0,e^{15},e^{25},0,e^{12})$	$-\frac{3}{5}m,-\frac{6}{5}m$	$3,3$

The real number m has to be a negative, in order for $\mathfrak{s}$ to be of Iwasawa type. Notice that the result holds just assuming non-degeneracy (thus dropping (3) in Definition 3.1).

In the present set-up, the Corollary matches to the result of [31] for appropriate choices of m. Moreover, the eigenvalue type is unique to each example, in contrast to the Einstein case where the solvmanifolds associated $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ and $\mathfrak{h}_3^{\mathbb{C}}$ have the same eigenvalue type.

As a by-product of the classification, the following necessary condition crops up:

4.5. **Corollary.** *With the above hypotheses, if Y has a G_2 structure of type $\mathcal{X}_4$, then N is either a 2-step nilpotent Lie group or a torus.* □

This is reflected in the fact that the metrics supported by these solvmanifolds arise on torus bundles over tori of various dimensions and rank, cf. §5. Notice that the only 2-step nilmanifold missing, so to speak, is that corresponding to $(0,0,0,0,e^{12},e^{34}) \cong \mathfrak{h}_3 \oplus \mathfrak{h}_3$. It is known to the authors that this Lie algebra admits a large family of half-flat $SU(3)$-structures. The product of the corresponding nilmanifold with some real interval can be endowed with a metric with holonomy contained in G_2 [21], but by the Theorem such metric will not be conformally equivalent to a homogeneous one on a solvable extension of $\mathcal{H}^3 \times \mathcal{H}^3$.

4.1. **Some consequences.** A slight change of approach allows to detect properties in a simpler way. Let

$$\alpha^1 = e^1 + ie^4, \quad \alpha^2 = e^2 - ie^3, \quad \alpha^3 = e^5 + ie^6$$

be the basis of complex $(1,0)$-forms determined by (2.1), so one may write

$$\Psi = \alpha^1 \wedge \alpha^2 \wedge \alpha^3 = \alpha^{123}.$$

Translate all $SU(3)$ forms into this language

$$\omega = -\tfrac{1}{2i}(\alpha^{1\bar{1}} + \alpha^{2\bar{2}} + \alpha^{3\bar{3}}) \quad \text{and} \quad \psi^+ = \tfrac{1}{2}(\alpha^{123} + \alpha^{\bar{1}\bar{2}\bar{3}}) \quad \textit{etc.},$$

with $\alpha^{\bar{\imath}}$ indicating the conjugate $\overline{\alpha^{\imath}}$ and α^{ij} standing for $\alpha^i\alpha^j,\ i,j=1,2,3$. Equations (4.2) become

$$\begin{aligned}2i\hat{d}(\alpha^{1\bar{1}}+\alpha^{2\bar{2}}+\alpha^{3\bar{3}}) &= (c_4-c_1)(\alpha^{123}+\alpha^{\bar{1}\bar{2}\bar{3}})+(c_3-c_2)(\alpha^{12\bar{3}}+\alpha^{\bar{1}\bar{2}3})+\\ &\quad (c_6-c_5)(\alpha^{1\bar{2}3}+\alpha^{\bar{1}2\bar{3}})\ -(6m+\textstyle\sum_i c_i)(\alpha^{\bar{1}23}+\alpha^{1\bar{2}\bar{3}}),\end{aligned}$$

$$\begin{aligned}-4i\hat{d}\alpha^{123} &= (2m-c_1-c_2-c_3-c_4)\alpha^{1\bar{1}2\bar{2}}\ +(2m-c_3-c_2-c_5-c_6)\alpha^{2\bar{2}3\bar{3}}-\\ &\quad (2m+c_1+c_4+c_5+c_6)\alpha^{1\bar{1}3\bar{3}}.\end{aligned}$$

Because of type, the terms in the latter can be treated separately

$$\left\{\begin{array}{lcl}4id\alpha^3 &=& (2m-c_1-c_2-c_3-c_4)\alpha^{\bar{1}\bar{2}}\\ 4id\alpha^2 &=& (2m+c_1+c_4+c_5+c_6)\alpha^{\bar{1}\bar{3}}\\ 4id\alpha^1 &=& (2m-c_2-c_3-c_5-c_6)\alpha^{\bar{2}\bar{3}}.\end{array}\right. \tag{4.10}$$

The special case in which J is actually a complex structure is instructive. Since the six-dimensional manifold is half-flat, hence has intrinsic torsion only in $\mathcal{W}_1^-,\mathcal{W}_2^-$ and $\mathcal{W}_3$, the further requirement that $\hat{d}\psi^-=0$ forces N to become balanced. By (4.2) the extension's coefficients satisfy the relation

$$c_1+c_4=c_5+c_6=-\tfrac{1}{3}(c_2+c_3).$$

In relation to the structures (4.3) – (4.9), equations (2.2) confirm that if (N,J) is Hermitian then $Hol(g)$ is a subgroup of G_2

4.6. **Proposition.** *On the solvable group corresponding to $\mathfrak{s}=\mathfrak{n}\oplus\mathbb{R}e_7$ the G_2 structure $\varphi=\omega e^7+\psi^+$ cannot be conformally parallel if the almost complex structure J on $\mathfrak{n}$ is integrable.* □

By this result, if one fixes the almost complex structure J, there exists a unique choice of ψ^+ in the $U(1)$-family of stable real 3-forms, since the closure of ψ^- renders J necessarily integrable.

Remark 4.7. A reasonable question is to ask whether this geometry has the potential to produce strong G_2-metrics [13]. As the torsion form is merely $\Phi=m\psi^-$, its closure entails that N is again a complex manifold (hence balanced). Although this is enough to conclude that the holonomy of Y reduces, things get even worse, for $d\psi^-$ has components in $\Lambda^3\mathfrak{n}^*\wedge e^7$ as well, forcing

$$c_1=c_4,c_5=c_6,c_2=c_3=-(c_1+c_5).$$

This gives $\hat{d}\omega=-3m\psi^+\in\mathcal{W}_1$, whence N has to be Kähler, confirming that the only solutions come from taking $m=0$ in (2.2). We conclude that if $dT=0$, the Lie algebra structure (3.3) simplifies to $de^j=\hat{d}e^j, de^7=0$, so we are merely looking at Y as the Riemannian product of N with $\mathbb{R}$, much of which is known [11].

A similar argument also restricts the range of m in the general set-up. We claim in fact that

$$2m\in\{c_1+c_2+c_3+c_4,\ -(c_1+c_4+c_5+c_6),\ c_2+c_3+c_5+c_6\}.$$

If $\mathfrak{s}$ does not satisfy the above relation, then all coefficients in (4.10) are different from zero, affecting the topology of N. Considering α^2 for instance, one sees that e^2 and e^3 cannot be simultaneously closed, so the manifold N cannot admit more than three independent closed

1-forms. But Theorem 4.3 tells that only Lie algebras with first Betti number $b_1 \geqslant 3$ crop up, and the unique 2-step algebra attaining the minimum is (4.9), which fails to satisfy the assumption.

The conditions to have a compatible almost Kähler structure are found in a similar fashion. It is non-obvious, and certainly unusual, that the symplectic condition also annihilates the component of the intrinsic torsion in $\mathcal{W}_2^-$, a module not directly depending upon $d\omega$:

4.8. **Proposition.** *Under the above assumptions, the nilpotent Lie group (N, J, ω) is symplectic only when it is a torus, in other words*

$$d\omega = 0 \iff \mathfrak{n} \text{ is Abelian.}$$

Proof. If ω is closed, its expression in complex form easily gives $c_1 = c_4, c_3 = c_2, c_5 = c_6$, which corresponds precisely to $J ad_{e_7} = ad_{e_7} J$; concerning Remark 3.2, it is definitely worth noticing that the almost complex structure and the derivation $D = ad_{e_7}$ commute just for two nilpotent Lie algebras, that is the Abelian one and the Iwasawa Lie algebra. The latter though does not satisfy the requirement that $m = -\sum c_i = -tr\, ad_{e_7}$, whence only the torus T^6 has symplectic structures generating a G_2-manifold of type $\mathcal{X}_4$. $\square$

5. Description of the Ricci-flat metrics

So $\mathfrak{s} = \mathfrak{n} \oplus \mathbb{R} e_7$ possesses a conformally parallel G_2 structure determined by the Lie types (4.3) – (4.9) of $\mathfrak{n}$. A transformation $g \mapsto e^{2f} g$ with conformal factor $df = -me^7$ produces Ricci-flat metrics, which we describe in detail. Since the corresponding simply-connected solvable Lie group Lie group S is diffeomorphic to $\mathbb{R}^7$, it is possible to find global coordinates $(x_1, \dots x_6, t)$ that describe the left-invariant 1-forms $e^1, \dots, e^6$ and $e^7 = dt$, all of which depend upon one real parameter $m \neq 0$. The general form for these metrics will thus be $g = e^{-2mt} \sum_{i=1}^{7} (e^i)^2$, and the explicit calculations will be relevant in the determination of holonomy groups.

In some cases, the solution can be related to the results of [17], whose metrics depend upon a function accounting for the scaling symmetry. We can thus prove that all our metrics admit a homothetic Killing field, i.e. a vector field Z such that $\mathcal{L}_Z g = cg, c \in \mathbb{R}$.

5.1. **The Abelian case.** For the Lie algebra (4.3), the coordinates

$$\begin{cases} e^i = e^{mt} dx_i, \quad i = 1, \dots, 6, \\ e^7 = dt \end{cases}$$

just yield the flat metric $g = \sum_{i=1}^{6} dx_i^2 + e^{-2mt} dt^2$ on $T^6 \times \mathbb{R}$.

5.2. **The algebra $\mathbb{R}^3 \oplus \mathfrak{h}_3$.** Consider the solvable extension of the product (4.4) of a torus T^3 with $\mathcal{H}^3$, and let

$$\begin{cases} e^i = e^{\frac{2}{3}mt} dx_i, \quad i = 1, 5, \\ e^l = e^{mt} dx_l, \quad l = 2, 4, 6, \\ e^3 = -\frac{2}{3} m\, e^{\frac{4}{3}mt} (dx_3 + x_5 dx_1), \\ e^7 = dt. \end{cases} \tag{5.1}$$

The Riemannian structure

$$g = dx_2^2 + dx_4^2 + dx_6^2 + e^{-\frac{2}{3}mt}(dx_1^2 + dx_5^2) + \tfrac{4}{9} m^2 e^{\frac{2}{3}mt} (dx_3 + x_5 dx_1)^2 + e^{-2mt} dt^2 \tag{5.2}$$

restricts to a special holonomy metric ds on $span\{x_1, x_3, x_5, t\}$ viewed as $Q \times \mathbb{R}$, Q being the total space of a circle bundle over T^2. In fact the subgroup of G_2 preserving ds is orthogonal, hence $Hol(g) = G_2 \cap SO(4) = SU(2)$.

5.3. **The algebra $(0, 0, e^{15} + e^{64}, 0, 0, 0)$.** When S corresponds to (4.5), we set

$$\begin{cases} e^i = e^{\frac{3}{4}mt} dx_i, \quad i = 1, 4, 5, 6, \\ e^2 = e^{mt} dx_2, \\ e^3 = -\frac{1}{2} m e^{\frac{3}{2}mt} (\frac{3}{2} dx_3 + x_5 dx_1 + x_4 dx_6), \\ e^7 = dt, \end{cases}$$

whence

$$g = dx_2^2 + e^{-\frac{1}{2}mt}(dx_1^2 + dx_4^2 + dx_5^2 + dx_6^2) + \frac{9}{16} m^2 e^{mt} (dx_3 + \tfrac{2}{3} x_5 dx_1 + \tfrac{2}{3} x_4 dx_6)^2 + e^{-2mt} dt^2 \tag{5.3}$$

has holonomy $SU(3) \subset G_2$. Restricting ourselves to $\langle x_2 \rangle^\perp$, we obtain a metric on the product of a principal T^1-bundle over T^4 with $\mathbb{R}$.

5.4. **The algebra $(0, e^{45}, e^{64} + e^{51}, 0, 0, 0)$.** Let us look at (4.6) now:

$$\begin{cases} e^i = e^{\frac{4}{5}mt} dx_i, \quad i = 1, 6, \\ e^2 = -\frac{3}{5} m\, e^{\frac{6}{5}mt} (dx_2 + \frac{2}{3} x_4 dx_5), \\ e^3 = -\frac{3}{5} m\, e^{\frac{7}{5}mt} (dx_3 - \frac{2}{3} x_1 dx_5 + \frac{2}{3} x_4 dx_6), \\ e^l = e^{\frac{3}{5}mt} dx_l, \quad l = 4, 5, \\ e^7 = dt \end{cases}$$

allow to write down a previously unknown exceptional metric

5.1. **Proposition.** *Let $\mathfrak{n}$ be the nilpotent Lie algebra defined by*

$$e_2 = [e_5, e_4], \quad e_3 = [e_6, e_4] = [e_1, e_5].$$

Then with the above conventions, the solvmanifold S relative to $\mathfrak{s} = \mathfrak{n} \oplus \mathbb{R}e_7$ carries a Riemannian metric

$$g = e^{-2mt} dt^2 + e^{-\frac{2}{5}mt}(dx_1^2 + dx_6^2) + e^{-\frac{4}{5}mt}(dx_4^2 + dx_5^2) + \frac{9}{25} m^2 e^{\frac{4}{5}mt}(dx_3 - \tfrac{2}{3} x_1 dx_5 + \tfrac{2}{3} x_4 dx_6)^2 + \frac{9}{25} m^2 e^{\frac{2}{5}mt} (dx_2 + \tfrac{2}{3} x_4 dx_5)^2 \tag{5.4}$$

whose holonomy group is precisely G_2. □

In sufficiently small neighbourhoods, this metric is clearly isometric to

$$ds^2 = V^3 dy^2 + V(dz_1^2 + dz_6^2) + V^2(dz_4^2 + dz_5^2) + V^{-2}\big(dz_3 + k(-z_1 dz_5 + z_4 dz_6)\big)^2 + V^2(dz_2 + k z_4 dz_5)^2,$$

with $V = ky$, on the product of $\mathbb{R}$ with a T^2-bundle over a T^4. The fibre coordinates are z_2, z_3, whilst y accounts for the $\mathbb{R}$ factor. The metric (5.4) has a symmetry generated by the homothetic Killing field

$$Z = -\tfrac{5}{m}\tfrac{\partial}{\partial t} + 4x_1\tfrac{\partial}{\partial x_1} + 4x_6\tfrac{\partial}{\partial x_6} + 3x_4\tfrac{\partial}{\partial x_4} + 3x_5\tfrac{\partial}{\partial x_5} + \tfrac{21}{5}mx_3\tfrac{\partial}{\partial x_3} + \tfrac{18}{5}mx_2\tfrac{\partial}{\partial x_2},$$

found by imposing invariance under a suitable scaling factor. For appropriate Killing vector fields, this feature is common to all other metrics in this section [17]. Note that Z does not correspond to e_7, for

$$(dZ^\flat)(\tfrac{\partial}{\partial t},\tfrac{\partial}{\partial x_1}) = \tfrac{\partial}{\partial t}Z^\flat(\tfrac{\partial}{\partial x_1}) - \tfrac{\partial}{\partial x_1}Z^\flat(\tfrac{\partial}{\partial t}) - Z^\flat([\tfrac{\partial}{\partial t},\tfrac{\partial}{\partial x_1}]) \neq 0.$$

5.5. **The algebra $(0, e^{45}, e^{46}, 0, 0, 0)$.** The coordinates

$$\begin{cases} e^1 = e^{mt}dx_1, & e^4 = e^{\frac{1}{2}mt}dx_4, \\ e^2 = \frac{1}{2}me^{\frac{5}{4}mt}(-\frac{3}{2}dx_2 + x_5dx_4), & e^i = e^{\frac{3}{4}mt}dx_i, \quad i = 5, 6, \\ e^3 = \frac{1}{2}me^{\frac{5}{4}mt}(-\frac{3}{2}dx_3 + x_6dx_4), & e^7 = dt, \end{cases}$$

relative to (4.7) produce the metric

$$g = dx_1^2 + \tfrac{9}{16}m^2e^{\frac{1}{2}mt}(dx_2 - \tfrac{2}{3}x_5dx_4)^2 + \tfrac{9}{16}m^2e^{\frac{1}{2}mt}(dx_3 - \tfrac{2}{3}x_6dx_4)^2 \\ + e^{-mt}dx_4^2 + e^{-\frac{1}{2}mt}(dx_5^2 + dx_6^2) + e^{-2mt}dt^2. \tag{5.5}$$

It has holonomy $SU(3)$, too. This can be recovered by looking at $\langle x_1\rangle^\perp = \mathbb{R}^6$, and the induced metric on the product of $\mathbb{R}$ with a principal T^2-bundle over T^3.

5.6. **The Iwasawa algebra.** The algebra (4.8) comes equipped with 1-forms

$$\begin{cases} e^i = e^{\frac{2}{3}mt}dx_i, \quad i = 1, 4, 5, 6, \\ e^2 = \frac{m}{3}e^{\frac{4}{3}mt}(2dx_2 + x_6dx_1 - x_4dx_5), \\ e^3 = -\frac{m}{3}e^{\frac{4}{3}mt}(2dx_3 + x_5dx_1 + x_4dx_6), \\ e^7 = dt. \end{cases} \tag{5.6}$$

Then

$$g = \tfrac{4}{9}m^2e^{\frac{2}{3}mt}(dx_3 + \tfrac{1}{2}x_5dx_1 + \tfrac{1}{2}x_4dx_6)^2 + \tfrac{4}{9}m^2e^{\frac{2}{3}mt}(dx_2 + \tfrac{1}{2}x_6dx_1 - \tfrac{1}{2}x_4dx_5)^2 + \\ e^{-\frac{2}{3}mt}(dx_1^2 + dx_4^2 + dx_5^2 + dx_6^2) + e^{-2mt}dt^2 \tag{5.7}$$

has holonomy G_2.

5.7. **The algebra $(0,0,e^{15},e^{25},0,e^{12})$.** Eventually,

$$g = e^{-\frac{4}{5}mt}(dx_1^2 + dx_2^2 + dx_5^2) + \tfrac{9}{25}m^2 e^{\frac{2}{5}mt}(dx_3 + \tfrac{2}{3}x_5 dx_1)^2 + \tfrac{9}{25}m^2 e^{\frac{2}{5}mt}(dx_4 - \tfrac{2}{3}x_2 dx_5)^2 + \tfrac{9}{25}m^2 e^{\frac{2}{5}mt}(dx_6 + \tfrac{2}{3}x_2 dx_1)^2 + e^{-2mt}dt^2 \tag{5.8}$$

belongs to (4.9), by means of

$$\begin{cases} e^i = e^{\frac{3}{5}mt}dx_i, \qquad i = 1,2,5, \\ e^3 = -\frac{1}{5}me^{\frac{6}{5}mt}(3dx_3 + 2x_5dx_1), \\ e^4 = -\frac{1}{5}me^{\frac{6}{5}mt}(3dx_4 - 2x_2dx_5), \\ e^6 = -\frac{1}{5}me^{\frac{6}{5}mt}(3dx_6 + 2x_2dx_1), \\ e^7 = dt. \end{cases}$$

These expressions identify g as a G_2 holonomy metric on $\mathbb{R}$ times a T^3-bundle over T^3.

Leaving the torus' flat structure aside, three of the metrics found have reduced holonomy, i.e. $SU(2)$ and $SU(3)$. These are attached to the algebras of Theorem 4.3 containing an Abelian summand

$$\mathbb{R}^3 \oplus \mathfrak{h}_3, \qquad \mathbb{R} \oplus \mathfrak{h}', \qquad \mathbb{R} \oplus \mathfrak{h}''$$

for given $\mathfrak{h}', \mathfrak{h}''$. This hints that the G_2 metrics could be reduced to lower dimensional structures of special type, with the same philosophy pursued in [3]. On the other hand (5.4), (5.7) and (5.8) are proper holonomy G_2 metrics, and are indeed built from algebras with an irreducible and more complicated structure. All metrics are scale-invariant. This is because the corresponding groups S can be decomposed into irreducible de Rham factors which are scale-invariant. The correspondence between $\mathfrak{n}$ and the holonomy of the Ricci-flat metric g supported by its rank-one solvable extension $\mathfrak{s}$ is summarised in the table.

Nilpotent algebra $\mathfrak{n}$	Holonomy
$(0,0,e^{15},0,0,0)$	$SU(2)$
$(0,0,e^{15}+e^{64},0,0,0)$	$SU(3)$
$(0,e^{45},e^{64}+e^{51},0,0,0)$	G_2
$(0,e^{45},e^{46},0,0,0)$	$SU(3)$
$(0,e^{16}+e^{45},e^{15}+e^{64},0,0,0)$	G_2
$(0,0,e^{15},e^{25},0,e^{12})$	G_2

6. Evolving the nilpotent $SU(3)$ structure

Let us turn to $\Gamma\backslash N \times \mathbb{R}$ and consider on the nilmanifold $\Gamma\backslash N$ the $SU(3)$ structure induced by that of N. In this section we wish to explain how one can use the so-called evolution equations discovered in [21]. These predict the deformation in time of special kinds of $SU(3)$ structures and their ability to give rise to metrics with holonomy contained in G_2. As a matter of fact, it turns out that half-flat $SU(3)$-manifolds represent the natural class with

the potential to evolve along the flow of the differential system and be preserved by it at the same time. We thus assume that $\omega(\mathrm{T}), \psi^+(\mathrm{T})$ is an $SU(3)$ structure depending on a locally defined real parameter $\mathrm{T} \in \mathbb{R}$. We may then regard the resulting 7-manifold as fibring over an interval, which accounts for a 'dynamic' inclusion of $SU(3)$ in the exceptional group. The central point is that the fundamental forms evolve according to the differential equations

$$\begin{cases} \hat{d}\omega = \dfrac{\partial \psi^+}{\partial \mathrm{T}}, \\ \hat{d}\psi^- = -\omega \wedge \dfrac{\partial \omega}{\partial \mathrm{T}}. \end{cases} \tag{6.1}$$

The compatibility relations restraining the almost Hermitian structure

$$\omega \wedge \psi^+ = 0, \tag{6.2}$$

$$\psi^+ \wedge \psi^- = \tfrac{2}{3}\omega^3 \tag{6.3}$$

are preserved in time. Considering the general difficulty in solving system (6.1), half-flatness provides the simplest examples of evolution structures — other than nearly Kähler ones. Besides, as all data in question is analytic, the solution is uniquely determined, hence one can expect the outcome to resemble one of the metrics of the previous section

6.1. **Proposition.** *Any of the Ricci-flat metrics on the solvable Lie group S with structure equations* (4.4) – (4.9)*, can be obtained evolving the $SU(3)$ structure on the* 2*-step nilmanifold* $\Gamma\backslash N$.

For understandable reasons the discussion will omit the case of T^6, for which the results of this section hold anyway, if trivially. To avoid being too long, the proof will rely on the detailed description of the technique for $\mathfrak{h}_3 \oplus \mathbb{R}^3$ and the Iwasawa algebra only. The forms $\psi^+(0),\ \omega^2(0)$ will be indicated by $\psi_0^+,\ \omega_0^2$.

First example. We begin by considering the nilpotent Lie algebra

$$(0, 0, \tfrac{2}{3}me^{15}, 0, 0, 0)$$

underlying that of (4.4). The forms defining the structure on $\Gamma\backslash N$ are deformed by means of exact elements in Chevalley–Eilenberg's cohomology

$$\Omega^3_{\mathrm{exact}} = \langle e^{125}, e^{145}, e^{156}\rangle, \quad \Omega^4_{\mathrm{exact}} = \langle e^{1245}, e^{1256}, e^{1456}\rangle.$$

We introduce the deformation functions, all depending upon T

$$\begin{cases} \frac{1}{2}\omega^2(\mathrm{T}) = P(\mathrm{T})\frac{1}{2}\omega_0^2 + D(\mathrm{T})e^{1245} + E(\mathrm{T})e^{1256} + F(\mathrm{T})e^{1456}, \\ \psi^+(\mathrm{T}) = Q(\mathrm{T})\psi_0^+ + A(\mathrm{T})e^{125} + B(\mathrm{T})e^{145} + C(\mathrm{T})e^{156}. \end{cases}$$

By asking $P(0) = Q(0) = 1,\ D(0) = E(0) = F(0) = A(0) = B(0) = C(0) = 0$, one is able to regain the initial structure (2.1) at time $\mathrm{T} = 0$. The expression for $\omega^2(\mathrm{T})$ suggests that the Kähler form, uniquely determined up to sign, must be of the following kind

$$\omega(\mathrm{T}) = x(\mathrm{T})e^{14} + y(\mathrm{T})e^{23} + z(\mathrm{T})e^{56} + w(\mathrm{T})e^{25} + j(\mathrm{T})e^{12},$$

for certain functions satisfying $x(0) = z(0) = -y(0) = 1,\ w(0) = j(0) = 0$. In the following, the explicit dependence upon T will be dropped, with the convention that the relations hold for all appropriate values of time. By computing $\omega \wedge \omega$ one finds the relations

$$zx = P + F, \quad xy = yz = -P, \quad xw = -D, \quad zj = E. \tag{6.4}$$

In addition, the primitivity of $\psi^\pm$ underlying equation (6.2) implies $yB = jQ,\ yC = wQ$. The first differential equation of (6.1) compares

$$\tfrac{\partial\psi^+}{\partial \mathrm{T}} = (Q' + A')e^{125} + Q'(-e^{345} + e^{136} + e^{246}) + B'e^{145} + C'e^{156}$$

with

$$\hat{d}\omega = \tfrac{2}{3}mye^{125},$$

dashed letters denoting derivatives with respect to T. This gives $Q(\mathrm{T}) = 1, B(\mathrm{T}) = C(\mathrm{T}) = 0$ for all T's, and $A'(\mathrm{T}) = \tfrac{2}{3}my(\mathrm{T})$. Therefore

$$\psi^+ = (A + 1)e^{125} - e^{345} + e^{136} + e^{246}.$$

From (6.4) one has $j(\mathrm{T}) = w(\mathrm{T}) = 0$, hence $D(\mathrm{T}) = E(\mathrm{T}) = 0$. The determination of an orthonormal basis of 1-forms enables one to get $\psi^-(\mathrm{T})$. Inspired by equations (5.1), one defines

$$\lambda^a e^1,\ \lambda^b e^2,\ \lambda^c e^3,\ \lambda^b e^4,\ \lambda^a e^5,\ \lambda^b e^6,$$

for some non-zero function $\lambda = \lambda(\mathrm{T})$ with $\lambda(0) = 1$. For $\psi^+ = \lambda^{2a+b}e^{125} + \lambda^{a+b+c}(-e^{345} + e^{136}) + \lambda^{3b}e^{246}$ to resemble the previous expression one takes $a + b + c = 0 = 3b$. One of many possible choices is $a = \frac{1}{2} = -c$, so that now $\psi^\pm$ assume the form

$$\psi^+ = \lambda e^{125} - e^{345} + e^{136} + e^{246}, \quad \psi^- = \lambda^{1/2}(e^{126} - e^{245} - e^{135}) - \lambda^{-1/2}e^{346}$$

and thus $\lambda(\mathrm{T}) = A(\mathrm{T}) + 1$. The second evolution equation

$$-(P' + F')e^{1456} + P'(e^{1423} + e^{2356}) = -\omega\omega' = \hat{d}\psi^- = \tfrac{2m}{3\sqrt{\lambda}}e^{1456}$$

implies $P(\mathrm{T}) = 1$ and $F' = -\frac{2m}{3\sqrt{A+1}}$. Eventually, volume normalisation (6.3) says that

$$\begin{cases} \sqrt{A + 1} = -\tfrac{2}{3}m(A')^{-1} \\ A(0) = 0. \end{cases}$$

The solution to this initial value problem reads

$$A(t) = (1 - m\mathrm{T})^{2/3} - 1,$$

so the geometric structure is evolving according to

$$\begin{aligned} \psi^+(\mathrm{T}) &= \sqrt[3]{(1 - m\mathrm{T})^2}\, e^{125} - e^{345} + e^{136} + e^{246}, \\ \omega(\mathrm{T}) &= \sqrt[3]{1 - m\mathrm{T}}\, (e^{14} + e^{56}) - \frac{1}{\sqrt[3]{1 - m\mathrm{T}}}\, e^{23}. \end{aligned}$$

The associated metric

$$g = (1 - m\mathrm{T})^{2/3}\big((e^1)^2 + (e^5)^2\big) + \sum_{i=2,4,6} (e^i)^2 + (1 - m\mathrm{T})^{-2/3}(e^3)^2 + d\mathrm{T}^2$$

mirrors precisely (5.2): the appropriate coordinate system $\{x_i\}$ on $\mathbb{R}^6$ is given by

$$e^i = dx_i, \quad i \neq 3, \qquad e^3 = -\tfrac{2}{3}m(dx_3 + x_5 dx_1),$$

and the correspondence follows once one identifies $1 - m\mathrm{T}$ with the conformal factor e^{-mt}.

Second example. Let us look at the Iwasawa Lie algebra

$$\big(0,-\tfrac{1}{3}m(e^{16}+e^{45}),\tfrac{1}{3}m(e^{15}-e^{46}),0,0,0\big),$$

whose evolution also deserves a detailed description. To start with, the deformation is given by

$$\begin{aligned}&\psi^+=Q\psi_0^+ +Ae^{145}+Be^{146}+Ce^{156}+D^{456},\\ &\tfrac{1}{2}\omega^2=\tfrac{1}{2}P\omega_0^2+Fe^{1456}+G(e^{1435}-e^{1426})+H(-e^{1436}-e^{1425})+\\ &\qquad\qquad L(e^{1356}+e^{2456})+M(e^{3456}-e^{1256}),\end{aligned}$$

the latter telling that the Kähler form is

$$\omega=xe^{14}+ye^{23}+ze^{56}+u(e^{35}-e^{26})+v(-e^{36}-e^{25})+\rho(e^{13}+e^{24})+\sigma(e^{34}-e^{12}).$$

This first relations obtained by comparison are

$$\begin{gathered}-P=xy=yz,\quad P+F=xz,\\ G=xu,\quad H=xv,\quad L=z\rho,\quad M=z\sigma,\\ u\rho+v\sigma=0,\quad -u\sigma+v\rho=0.\end{gathered}\tag{6.5}$$

Then the evolution of ψ^+ immediately annihilates A,B,C,D and yields

$$Q'=\tfrac{1}{3}my,$$

as ψ_0^+ is exact. On the other hand $0=\frac{1}{Q}\omega\psi^+=-2e^{23}(ue^{456}+ve^{156}+\rho e^{146}-\sigma e^{145})$ forces the vanishing of most of the remaining coefficients

$$\psi^+=Q\psi_0^+,\qquad \tfrac{1}{2}\omega^2=\tfrac{1}{2}P\omega_0^2+Fe^{1456}.$$

The orthonormal basis $\lambda^a e^i,\ i=1,4,5,6,\ \lambda^b e^j,\ j=2,3$ is chosen in order to mimic equations (5.6). The actual values of $a,b\in\mathbb{R}$ are irrelevant at present, one could for example take $a=-1,b=1$. In any case

$$\psi^+=\lambda^{2a+b}\psi_0^+,\quad \psi^-=\lambda^{2a+b}\psi_0^-,\ \text{with } \lambda^{2a+b}=Q.$$

From $\hat{d}\psi^-=-\omega\omega'$ one obtains $P=1$ and $F'=\frac{4}{3}mQ$, so that each of ω^2,ψ^+ evolves in one direction only. This could have been predicted by counting dimensions, see [24]. The first line in (6.5) tells that $x=z=-1/y=\sqrt{F+1}$, and (6.3) gives the quartic curve

$$Q^2=\sqrt{F+1},$$

so the evolution equations are equivalent to the first order system

$$\begin{cases}Q'(\mathrm{T})=-\frac{m}{3}Q^{-2}\\ F'(\mathrm{T})=-\frac{4}{3}m(F+1)^{1/4}\\ Q(0)=1,\quad F(0)=0.\end{cases}$$

The first equation is solved by $Q(\mathrm{T})=(1-m\mathrm{T})^{1/3}$, hence $F(\mathrm{T})=(1-m\mathrm{T})^{4/3}-1$. Eventually, the $SU(3)$ structure results in

$$\begin{aligned}\psi^+(\mathrm{T})&=(1-m\mathrm{T})^{1/3}\,(e^{125}-e^{345}+e^{136}+e^{246}),\\ \omega(\mathrm{T})&=(1-m\mathrm{T})^{2/3}\,(e^{14}+e^{56})-\frac{1}{(1-m\mathrm{T})^{2/3}}\,e^{23}.\end{aligned}$$

These data identify the non-integrable complex structure $-J_3$ studied in a broader context by [1] and the flow corresponds to the one given in [3, ex. 2 (iii)]. A glance at the level

curves of the background Hamiltonian function confirms that the almost complex structure degenerates at time $\mathrm{T} = 1/m$. The corresponding G_2-metric

$$g = (1 - m\mathrm{T})^{2/3} \sum_{\substack{i=1,4,\\5,6}} (e^i)^2 + \frac{1}{(1 - m\mathrm{T})^{2/3}} \left((e^2)^2 + (e^3)^2\right) + d\mathrm{T}^2$$

has its counterpart in (5.7) when $t = \frac{1}{m} \ln |1 - m\mathrm{T}|$.

The other cases. 1. With the same technique we tackle the 6-dimensional Lie algebra with non-trivial brackets $[e_5, e_1] = \frac{1}{2} m e_3 = [e_4, e_6]$, isomorphic to $(0, 0, 0, 0, 0, e^{12} + e^{34})$. As time goes by, the outcoming $SU(3)$ geometry is described by

$$\psi^+(\mathrm{T}) = \sqrt{1 - m\mathrm{T}}\,(e^{125} + e^{246}) + e^{136} - e^{345},$$
$$\omega(\mathrm{T}) = \sqrt{1 - m\mathrm{T}}\,(e^{14} + e^{56}) - \frac{1}{\sqrt{1 - m\mathrm{T}}} e^{23},$$

so the Riemannian structure is

$$g = \sqrt{1 - m\mathrm{T}} \sum_{\substack{j=1,4,\\5,6}} (e^j)^2 + (e^2)^2 + \frac{1}{1 - m\mathrm{T}} (e^3)^2 + d\mathrm{T}^2,$$

in agreement with (5.3).

2. We next apply the evolution machinery to $\left(0, -\frac{2}{5} m e^{45}, \frac{2}{5} m(e^{15} - e^{46}), 0, 0, 0\right)$. The four-form flows according to $\frac{1}{2}\omega^2(\mathrm{T}) = (1 - m\mathrm{T})^{6/5} e^{1456} - e^{1423} - e^{2356}$, whose square root provides

$$\omega(\mathrm{T}) = (1 - m\mathrm{T})^{3/5}(e^{14} + e^{56}) - (1 - m\mathrm{T})^{-3/5} e^{23},$$

while the 3-form is

$$\psi^+(\mathrm{T}) = (1 - m\mathrm{T})^{2/5}(-e^{345} + e^{125} + e^{246}) + e^{136}.$$

The Riemannian metric

$$\begin{aligned} g = (1 - m\mathrm{T})^{2/5} \left((e^1)^2 + (e^6)^2\right) + (1 - m\mathrm{T})^{4/5} \left((e^4)^2 + (e^5)^2\right) \\ + (1 - m\mathrm{T})^{-2/5} (e^2)^2 + (1 - m\mathrm{T})^{-4/5} (e^3)^2 + d\mathrm{T}^2, \end{aligned}$$

basically recovers that of Proposition 5.1.

3. The deformation of the Lie structure corresponding to $de^2 = -\frac{1}{5} m e^{45}$, $de^3 = -\frac{1}{5} m e^{46}$ is

$$\psi^+ = e^{125} + e^{136} + \sqrt{1 - m\mathrm{T}}\,(e^{246} - e^{345}),$$
$$\omega = \sqrt{1 - m\mathrm{T}}\,(e^{14} + e^{56}) - \frac{1}{\sqrt{1 - m\mathrm{T}}} e^{23},$$

hence we get

$$g = (e^1)^2 + \frac{1}{\sqrt{1 - m\mathrm{T}}} \left((e^2)^2 + (e^3)^2\right) + (1 - m\mathrm{T})\,(e^4)^2 + \sqrt{1 - m\mathrm{T}}\,\left((e^5)^2 + (e^6)^2\right) + d\mathrm{T}^2,$$

related to (5.5).

4. At last, let us write what happens to $(0,0,\frac{2}{5}me^{15},\frac{2}{5}me^{25},0,\frac{2}{5}me^{12})$. After some computations one finds that

$$\psi^+ = (1-m\mathrm{T})^{6/5}\, e^{125} + e^{136} + e^{246} - e^{345},$$
$$\omega = (1-m\mathrm{T})^{1/5}\,(e^{14} - e^{23} + e^{56})$$

are compatible with the metric

$$g = (1-m\mathrm{T})^{4/5} \sum_{i=1,2,5} (e^i)^2 + (1-m\mathrm{T})^{-2/5} \sum_{j=3,4,6} (e^j)^2 + d\mathrm{T}^2,$$

see (5.8). □

It is no coincidence that in all cases the identification between the 'evolved' holonomy metrics and the ones found in §5 is attained by uniformly putting $\mathrm{T} = 1/m(1-e^{-mt})$ and using global coordinates $x_1,\dots,x_6$ on the nilpotent Lie group N to represent the left-invariant forms $\{e^i\}$. Which brings to the completeness' properties of the metrics. We are always in presence of a unique singularity, determined by $f(t)=\exp(-mt)$ or, if one prefers, by the linear function $\tilde f(\mathrm{T}) = 1-m\mathrm{T}$. This means that away from the degeneration, all metrics are complete in one direction of time, i.e. the tensors g,ψ^+,ω describe a smooth structure for $\mathrm{T}\in(-\infty,\mathrm{T}_0]$, $\mathrm{T}_0 < 1/m$, see related discussion in [3].

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